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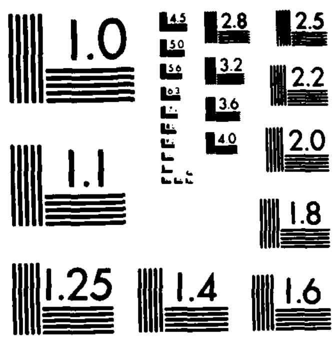

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Light Rays in a Spherically Symmetric Medium

by

Daniel T. Gillespie
Research Department

JUNE 1986

NAVAL WEAPONS CENTER
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We derive from Maxwell's equations a set of three coupled first-order differential equations for high-frequency light ray trajectories in a medium whose index of refraction $n(\mathbf{r})$ at any point $\mathbf{r}$ is a function only of r. The derived equations are simple and singularity-free when r is bounded away from zero, and should therefore be especially well suited to numerical ray tracing in spherically symmetric models of Earth's atmosphere. We obtain analytical solutions to these ray equations for two mathematically simple cases, and we discuss strategies for obtaining numerical solutions for more realistic cases.

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SUMMARY

We derive from Maxwell's equations a set of three coupled first-order differential equations for high-frequency light ray trajectories in a medium whose index of refraction $n(\mathbf{r})$ at any point $\mathbf{r}$ is a function only of r. The derived equations, displayed in Equations 33 or 34 and involving the variables shown in Figure 2, are simple and singularity-free when r is bounded away from zero. These equations should therefore be especially well suited to numerical ray tracing in spherically symmetric models of Earth's atmosphere. We obtain analytical solutions to these ray equations for two mathematically simple cases, and we discuss strategies for obtaining numerical solutions for more realistic cases.

MAXWELL'S EQUATIONS

In the absence of free electric charges and currents, Maxwell's equations read

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{1a}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1c}$$

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D}, \tag{1d}$$

where $\nabla \equiv \partial/\partial\mathbf{r}$ and $\partial_t \equiv \partial/\partial t$. In a linear, isotropic, stationary medium, we have the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu, \tag{2}$$

where ε and μ are scalars that are independent of t but *may* depend on $\mathbf{r}$. Substituting Equations 2 into Equations 1b and 1d, and noting that

$$\nabla \cdot (\varepsilon \mathbf{E}) = \varepsilon \nabla \cdot \mathbf{E} + \nabla\varepsilon \cdot \mathbf{E},$$

and

$$\nabla \times (\mathbf{B}/\mu) = \mu^{-1} \nabla \times \mathbf{B} + \nabla(\mu^{-1}) \times \mathbf{B}$$

we obtain

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{3a}$$

$$\nabla \cdot \mathbf{E} = -\varepsilon^{-1} \nabla \varepsilon \cdot \mathbf{E}, \tag{3b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3c}$$

$$\nabla \times \mathbf{B} = \mu\varepsilon\, \partial_t \mathbf{E} - \mu \nabla(\mu^{-1}) \times \mathbf{B}. \tag{3d}$$

Equations 3 define the **E** and **B** fields in any region that has no free electric charges or currents, and that is characterized by scalar point functions $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$.

PLANE WAVE SOLUTIONS FOR CONSTANT ε AND μ

In the special case in which ε and μ are independent of **r**, Maxwell's equations 3 evidently simplify to

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{4a}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{4b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4c}$$

$$\nabla \times \mathbf{B} = \mu\varepsilon\, \partial_t \mathbf{E}. \tag{4d}$$

We shall prove that these four coupled equations admit solutions of the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp(ik_0[n\hat{\mathbf{k}} \cdot \mathbf{r} - ct]), \tag{5a}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 \exp(ik_0[n\hat{\mathbf{k}} \cdot \mathbf{r} - ct]), \tag{5b}$$

where c is the speed of light in vacuum, and where the constant scalar n and the constant vectors $\mathbf{E}_0$, $\mathbf{B}_0$, and $\hat{\mathbf{k}}$ (a circumflex denotes a vector of unit length) satisfy the following three conditions:

$$n = c\,(\mu\varepsilon)^{1/2}; \tag{6a}$$

$$\{\hat{\mathbf{k}}, \mathbf{E}_0, \mathbf{B}_0\} \text{ is a right-orthogonal triad;} \tag{6b}$$

$$B_0 = nE_0/c. \tag{6c}$$

Before presenting the proof, let us observe that the functions in Equations 5 describe a plane wave disturbance that propagates in the direction $\hat{\mathbf{k}}$ with speed c/n. The wave is plane because the surfaces of constant phase or "wave fronts" are defined by $\hat{\mathbf{k}} \cdot \mathbf{r}$ = constant, which is the equation of a plane normal to $\hat{\mathbf{k}}$. The wave is evidently sinusoidal with period T and wavelength λ such that $k_0 c = 2\pi/T$ and $k_0 n = 2\pi/\lambda$; thus, the frequency $\nu = 1/T$ and wavelength λ of the sinusoidal plane wave described by Equations 5 are

$$\nu = k_0 c/2\pi, \tag{7a}$$

$$\lambda = 2\pi/k_0 n. \tag{7b}$$

To prove that Equations 5 satisfy Equations 4 if Equations 6 hold, we let ϕ denote the argument of the exponentials in Equations 5, and we note that the divergence and curl of $\mathbf{E}(\mathbf{r},t)$ have the form

$$\nabla * \mathbf{E}(\mathbf{r},t) = \nabla(e^{\phi}) * \mathbf{E}_0 = e^{\phi}\nabla\phi * \mathbf{E}_0,$$

where "$*$" denotes either "$\cdot$" or "$\times$" throughout. Since the gradient of ϕ is given by

$$\nabla\phi = ik_0 n\,\nabla(\hat{\mathbf{k}}\cdot\mathbf{r}) = ik_0 n\,\hat{\mathbf{k}},$$

then we get

$$\nabla * \mathbf{E}(\mathbf{r},t) = e^{\phi}(ik_0 n)\,\hat{\mathbf{k}} * \mathbf{E}_0.$$

It similarly follows that

$$\nabla * \mathbf{B}(\mathbf{r},t) = e^{\phi}(ik_0 n)\,\hat{\mathbf{k}} * \mathbf{B}_0.$$

Substituting these divergence and curl relations into Equations 4 and then dividing through by e^{ϕ}, we obtain

$$ik_0 n\,\mathbf{k}\times\mathbf{E}_0 = +ik_0 c\,\mathbf{B}_0, \quad (8a)$$

$$ik_0 n\,\mathbf{k}\cdot\mathbf{E}_0 = 0, \quad (8b)$$

$$ik_0 n\,\mathbf{k}\cdot\mathbf{B}_0 = 0, \quad (8c)$$

$$ik_0 n\,\mathbf{k}\times\mathbf{B}_0 = \mu\varepsilon(-ik_0 c)\,\mathbf{E}_0. \quad (8d)$$

The middle two equations show that $\hat{\mathbf{k}}$ must be perpendicular to both $\mathbf{E}_0$ and $\mathbf{B}_0$, and the first (or last) equation then shows that $\mathbf{E}_0$ and $\mathbf{B}_0$ must be perpendicular to each other, with $\hat{\mathbf{k}}$, $\mathbf{E}_0$, $\mathbf{B}_0$ forming a right-orthogonal triad, thus establishing Equation 6b. Given Equation 6b, the first and last of Equations 8 become

$$nE_0 = cB_0 \quad \text{and} \quad nB_0 = \mu\varepsilon c E_0,$$

respectively; together, these two equations imply Equations 6a and 6c.

DISTORTED PLANE WAVE SOLUTIONS FOR SPATIALLY DEPENDENT ε AND μ

If the scalars ε and μ that characterize the electromagnetic properties of the medium are functions of position $\mathbf{r}$, then we must use Maxwell's equations in the form of Equations 3 instead of Equations 4. Let us investigate the possibility of a solution to Equations 3 of the following form [cf. Equations 5]:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})\exp(\,ik_0[L(\mathbf{r}) - ct]\,), \quad (9a)$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\exp(\,ik_0[L(\mathbf{r}) - ct]\,). \quad (9b)$$

These two functions describe a *distorted plane wave* that propagates locally in the direction of $\nabla L(\mathbf{r})$ with speed $c/|\nabla L(\mathbf{r})|$. Notice that $\nabla L(\mathbf{r})$ is normal to the (generally nonplanar) surface $L(\mathbf{r}) =$ constant, the surface of constant phase or "wave front." At any fixed point $\mathbf{r}$ the wave is temporally sinusoidal with frequency ν satisfying $k_0 c = 2\pi\nu$. Although the wave is not in general spatially sinusoidal, we can ascribe to it a *local* wavelength $\lambda(\mathbf{r})$ satisfying $k_0|\nabla L(\mathbf{r})| = 2\pi/\lambda(\mathbf{r})$. Hence, the distorted plane waves in Equations 9 have frequency and local wavelength [cf. Equations 7]

$$\nu = k_0 c/2\pi, \tag{10a}$$

$$\lambda(\mathbf{r}) = 2\pi/k_0|\nabla L(\mathbf{r})|. \tag{10b}$$

To investigate whether $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ in Equations 9 can satisfy Maxwell's equations (3), we again let ϕ denote the phase of the waves, and we note that

$$\begin{aligned} \nabla * \mathbf{E}(\mathbf{r},t) &= \nabla(e^{\phi}) * \mathbf{E}_0 + e^{\phi}\nabla * \mathbf{E}_0 \\ &= e^{\phi}\nabla\phi * \mathbf{E}_0 + e^{\phi}\nabla * \mathbf{E}_0 \\ &= e^{\phi}(ik_0\nabla L) * \mathbf{E}_0 + e^{\phi}\nabla * \mathbf{E}_0, \end{aligned}$$

where again "$*$" denotes either "$\cdot$" or "$\times$" throughout. Substituting this and the analogous relation for $\nabla * \mathbf{B}(\mathbf{r},t)$ into Equations 3 and then dividing through by e^{ϕ}, we get

$$ik_0\nabla L \times \mathbf{E}_0 + \nabla \times \mathbf{E}_0 = +ik_0 c\, \mathbf{B}_0, \tag{11a}$$

$$ik_0\nabla L \cdot \mathbf{E}_0 + \nabla \cdot \mathbf{E}_0 = -\varepsilon^{-1}\nabla\varepsilon \cdot \mathbf{E}_0, \tag{11b}$$

$$ik_0\nabla L \cdot \mathbf{B}_0 + \nabla \cdot \mathbf{B}_0 = 0, \tag{11c}$$

$$ik_0\nabla L \times \mathbf{B}_0 + \nabla \times \mathbf{B}_0 = \mu\varepsilon(-ik_0 c)\mathbf{E}_0 - \mu\nabla(\mu^{-1}) \times \mathbf{B}_0. \tag{11d}$$

In general, then, if $\mathbf{E}_0(\mathbf{r})$, $\mathbf{B}_0(\mathbf{r})$, $L(\mathbf{r})$, and k_0 are chosen to satisfy Equations 11, the distorted plane waves of Equations 9 will satisfy Maxwell's equations. But now we consider the special case of *geometrical optics*, which corresponds to light waves of very high frequencies ν, or by Equations 10a, light waves with *very large* k_0. In that limit, we can evidently *approximate* Equations 11 by dropping all terms that do not contain a factor k_0. But Equations 11 thus approximated are seen to be *identical* to the plane wave Equations 8, *provided* we replace

$$\nabla L \rightarrow n\hat{\mathbf{k}}. \tag{12}$$

We have previously shown that satisfaction of Equations 8 requires satisfaction of Equations 6. Therefore, using Equation 12, we conclude that satisfaction of Equations 11 *in the limit of infinitely large* k_0 requires:

$$|\nabla L(\mathbf{r})| = c\,(\mu(\mathbf{r})\,\varepsilon(\mathbf{r}))^{1/2}; \tag{13a}$$

$$\{\nabla L(\mathbf{r}), \mathbf{E}_0(\mathbf{r}), \mathbf{B}_0(\mathbf{r})\} \text{ is a right-orthogonal triad;} \tag{13b}$$

$$B_0(\mathbf{r}) = |\nabla L(\mathbf{r})|\, E_0(\mathbf{r})/c. \tag{13c}$$

In summary, if Equations 13 are satisfied, then the distorted plane waves $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ in Equations 9 satisfy Maxwell's equations (3) in the limit $k_0 \rightarrow \infty$.

We define the *index of refraction n* as the scalar point function

$$n \equiv n(\mathbf{r}) \equiv c\,(\mu(\mathbf{r})\,\varepsilon(\mathbf{r}))^{1/2}. \tag{14}$$

This definition agrees with the plane wave equation (6a). The requirement in Equation 13a now takes the form

$$|\nabla L(\mathbf{r})| = n(\mathbf{r}). \tag{15}$$

This equation is called the *eikonal equation*, and is a key result of this section. As we have previously noted, the distorted plane waves in Equations 9 propagate with speed $c/|\nabla L(\mathbf{r})|$, so we may conclude from Equation 15 that $n(\mathbf{r})$ is the ratio of the propagation speed in a vacuum to the propagation speed in the medium at point $\mathbf{r}$. This interpretation of $n(\mathbf{r})$ is in agreement with the customary definition of the index of refraction.

We shall also define the *unit ray vector* $\hat{\mathbf{k}}$ as the vector point function

$$\hat{\mathbf{k}} \equiv \hat{\mathbf{k}}(\mathbf{r}) \equiv \nabla L(\mathbf{r})/|\nabla L(\mathbf{r})|. \tag{16}$$

In close analogy with the interpretation of $\hat{\mathbf{k}}$ in the plane wave case, $\hat{\mathbf{k}}(\mathbf{r})$ defines the direction of propagation of the distorted plane waves in Equations 9 at the point $\mathbf{r}$; equivalently, $\hat{\mathbf{k}}(\mathbf{r})$ is the unit tangent to the light ray at $\mathbf{r}$. With Equation 16, the eikonal equation (15) can be written

$$\nabla L(\mathbf{r}) = n(\mathbf{r})\hat{\mathbf{k}}(\mathbf{r}), \tag{17}$$

thus making formal the correspondence noted earlier in Equation 12.

EQUATION FOR THE TURNING RATE OF $\hat{\mathbf{k}}$

We shall now use the eikonal equation to derive an equation for the local turning rate of the unit ray vector $\hat{\mathbf{k}}$. To this end, we consider a distorted plane wave [cf. Equations 9] with eikonal function $L(\mathbf{r})$ in the neighborhood of some point P. Let $n(P)$ and $\nabla n(P)$ denote the respective values of the index of refraction and its gradient at P, and let $L(\mathrm{P})$ and $\nabla L(\mathrm{P})$ denote the respective values of the eikonal function and its gradient at P. The direction of propagation of the wave front at P, i.e., the tangent to the ray trajectory at P, is $\hat{\mathbf{k}}(\mathrm{P}) \equiv \nabla L(\mathrm{P})/|\nabla L(\mathrm{P})|$. Let $\hat{\mathbf{u}}$ be the unit vector in the plane of $\hat{\mathbf{k}}(\mathrm{P})$ and $\nabla n(\mathrm{P})$ that makes a *right* angle with $\hat{\mathbf{k}}(\mathrm{P})$ and an *acute* angle with $\nabla n(\mathrm{P})$, as shown in Figure 1. And let P′ be a point whose position vector relative to P is $\hat{\mathbf{u}}\mathrm{d}\rho$, where $\mathrm{d}\rho$ is a positive infinitesimal. Since $\hat{\mathbf{u}}$ by construction is normal to $\nabla L(\mathrm{P})$, then the smallness of $\mathrm{d}\rho$ ensures that every point on the line segment PP′ has $L(\mathbf{r}) = L(\mathrm{P})$; thus, the segment PP′ lies in the wave front through P.

Now let Q be a point whose position vector relative to P is $\hat{\mathbf{k}}(\mathrm{P})\mathrm{d}s$, where $\mathrm{d}s$ is another positive infinitesimal, also shown in Figure 1. As we move along the light ray the distance $\mathrm{d}s$ from P to Q, the value of the eikonal function $L(\mathbf{r})$ changes by an amount

$$\mathrm{d}L = \nabla L(\mathrm{P}) \cdot \hat{\mathbf{k}}(\mathrm{P})\mathrm{d}s = n(\mathrm{P})\hat{\mathbf{k}}(\mathrm{P}) \cdot \hat{\mathbf{k}}(\mathrm{P})\mathrm{d}s = n(\mathrm{P})\mathrm{d}s, \tag{18}$$

where the second equality has invoked the eikonal equation (17). Similarly, as we move an infinitesimal distance ds' in the ray direction $\hat{\mathbf{k}}(P') \approx \hat{\mathbf{k}}(P)$ from point P′ to some point Q′ (Figure 1), the eikonal function changes by an amount

$$dL' = n(P')ds. \tag{19}$$

Let us choose ds' so that

$$dL' = dL. \tag{20}$$

In that case, every point on the line segment QQ′ has $L(\mathbf{r}) = L(P) + dL$; thus, the segment QQ′ lies in the wave front through Q.

Given Equation 20, we may deduce from Equations 18 and 19 that

$$n(P)ds = n(P')ds'. \tag{21}$$

Since the angle between $\hat{\mathbf{u}}$ and $\nabla n(P)$ is acute, then $n(P') \geq n(P)$, so it follows from Equation 21 that $ds' \leq ds$. Evidently, in moving along the light ray the infinitesimal distance ds from P to Q, the wave front and its attendant unit normal have rotated through the infinitesimal angle

$$d\beta = \frac{(ds - ds')}{d\rho} = \frac{ds}{d\rho}\left(1 - \frac{ds'}{ds}\right), \tag{22}$$

the sense of this rotation being from $\mathbf{k}(P)$ towards $\hat{\mathbf{u}}$. Using Equation 21, Equation 22 becomes

$$d\beta = \frac{ds}{d\rho}\left(1 - \frac{n(P)}{n(P')}\right). \tag{23}$$

Now, to first order in $d\rho$ we have

$$n(P') = n(P) + \nabla n(P) \cdot \hat{\mathbf{u}} d\rho. \tag{24}$$

Substituting into Equation 23 gives

$$d\beta = \frac{ds}{d\rho}\left[1 - \left(1 + \frac{\nabla n(P) \cdot \hat{\mathbf{u}} d\rho}{n(P)}\right)^{-1}\right].$$

Again to first order in $d\rho$ we have

$$\left(1 + \frac{\nabla n(P) \cdot \hat{\mathbf{u}} d\rho}{n(P)}\right)^{-1} = 1 - \frac{\nabla n(P) \cdot \hat{\mathbf{u}} d\rho}{n(P)}.$$

Substituting into the previous expression and simplifying, we finally obtain

$$d\beta = \frac{\nabla n \cdot \hat{\mathbf{u}}}{n} \, ds, \tag{25}$$

where ∇n and n are henceforth understood to be evaluated at P.

Equation 25 shows that the rotation of $\hat{\mathbf{k}}$ towards $\hat{\mathbf{u}}$ is proportional to $\nabla n \cdot \hat{\mathbf{u}}$. If $\hat{\mathbf{v}}$ is another unit vector, perpendicular to *both* $\hat{\mathbf{k}}$ and $\hat{\mathbf{u}}$ (i.e., $\hat{\mathbf{v}} = \hat{\mathbf{k}} \times \hat{\mathbf{u}}$), then we can similarly show that the rotation of $\hat{\mathbf{k}}$ towards $\hat{\mathbf{v}}$ is proportional to $\nabla n \cdot \hat{\mathbf{v}}$. But if $\hat{\mathbf{v}}$ is perpendicular to both $\hat{\mathbf{k}}$ and $\hat{\mathbf{u}}$, then (Figure 1) $\hat{\mathbf{v}}$ must also be perpendicular to ∇n; therefore, $\nabla n \cdot \hat{\mathbf{v}} = 0$, and there is no rotation of $\hat{\mathbf{k}}$ toward $\hat{\mathbf{v}}$. In other words, $\hat{\mathbf{k}}$(Q) in Figure 1 indeed lies in the plane of $\hat{\mathbf{k}}$(P) and ∇n(P).

Equation 25 is the principal result of this section. This equation gives the infinitesimal angle $d\beta$ through which the unit ray vector $\hat{\mathbf{k}}$ will turn in the next infinitesimal distance ds along the ray. In interpreting Equation 25, we must remember that $\hat{\mathbf{u}}$ lies in the plane of $\hat{\mathbf{k}}$ and ∇n, making a right angle with $\hat{\mathbf{k}}$ and an acute angle with ∇n; furthermore, the positive sense of $d\beta$ is from $\hat{\mathbf{k}}$ toward $\hat{\mathbf{u}}$. The unit ray vector thus turns toward regions of higher n in the plane defined by itself and ∇n.

The *local turning rate* of the ray is formally defined to be $d\beta/ds$, and is immediately calculable from Equation 25. Figure 1 shows that $ds/d\beta$, the reciprocal of the local turning rate, is the *local radius of curvature* of the ray.

THE RAY EQUATIONS FOR $n = n(|\mathbf{r}|)$

We shall now use the equation for the turning rate of the unit ray vector, Equation 25, to derive a set of differential equations from which the ray trajectory can be calculated in the case that the index of refraction n at any point $\mathbf{r}$ is a function only of r. This case serves as a simplified model of Earth's atmosphere, with the origin at the center of the Earth and r bounded below by the radius of the Earth.

Let P be any point in the medium with position vector $\mathbf{r} = r\hat{\mathbf{r}}$ relative to the origin O, and let $\hat{\mathbf{k}}$ be the unit ray vector at P. See Figure 2. Let $\hat{\mathbf{z}}$ be any constant unit vector in the plane of $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$, and let θ be the angle between $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$. Finally, let $\hat{\boldsymbol{\theta}}$ be the unit vector in the plane of $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$ that is perpendicular to $\hat{\mathbf{r}}$ and in the direction of increasing θ, and let ε be the angle between $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{k}}$. We may regard ε as the "elevation angle" of $\hat{\mathbf{k}}$. The "state" of the ray can evidently be defined by the three variables r, θ and ε. Essentially, r and θ tell us where the ray is, while ε tells us where the ray is going. As shown in Figure 3, movement along the ray through an infinitesimal distance ds will bring us to a point Q whose position vector relative to O is

$$\mathbf{r} + d\mathbf{r} = \mathbf{r} + \hat{\mathbf{k}} ds.$$

Let N be the intersection of the line OQ and the line through P parallel to $\hat{\boldsymbol{\theta}}$. Since d$s$ is infinitesimally small, the triangle PQN may be regarded as a *right* triangle with hypotenuse PQ = ds and sides NQ = dr and PN = rdθ. Thus, from Figure 3 we have

$$\sin\varepsilon = (\mathrm{d}r)/(\mathrm{d}s) \quad \text{and} \quad \cos\varepsilon = (r\mathrm{d}\theta)/(\mathrm{d}s),$$

whence

$$\mathrm{d}r = \sin\varepsilon\, \mathrm{d}s, \tag{26}$$

and

$$\mathrm{d}\theta = r^{-1}\cos\varepsilon\, \mathrm{d}s. \tag{27}$$

Equations 26 and 27 give the respective changes in r and θ as we move an infinitesimal distance ds along the ray from P. We have only to calculate the concomitant change in ε. For this we must use the turning rate equation (25).

The assumption $n(\mathbf{r}) = n(r)$ implies that

$$\nabla n = n'(r)\hat{\mathbf{r}}. \tag{28}$$

In Figure 4 we show ∇n , along with the unit vector $\hat{\mathbf{u}}$ that lies in the plane of $\hat{\mathbf{k}}$ and ∇n making a right angle with $\hat{\mathbf{k}}$ and an acute angle with ∇n . We also show in Figure 4 the angle β that $\hat{\mathbf{k}}$ makes with a fixed direction in space; we have taken that fixed direction to be $-\hat{\mathbf{z}}$ in order to comply with the requirement that an increase in β correspond to a rotation of $\hat{\mathbf{k}}$ toward $\hat{\mathbf{u}}$ (See discussion following Equation 25.) Since the geometry of Figure 4 implies that the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{r}}$ is equal to ε, we have

$$\nabla n \cdot \hat{\mathbf{u}} = n'(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{u}} = n'(r)\cos\varepsilon.$$

Therefore, Equation 25 gives

$$\mathrm{d}\beta = [n'(r)/n(r)]\cos\varepsilon\, \mathrm{d}s. \tag{29}$$

From the geometry of the triangle in Figure 4 containing the two angles θ and β, it is seen that

$$\theta + \beta + (\pi/2 - \varepsilon) = \pi. \tag{30}$$

Therefore,

$$\varepsilon = \theta + \beta - \pi/2,$$

whence

$$\mathrm{d}\varepsilon = \mathrm{d}\theta + \mathrm{d}\beta. \tag{31}$$

Substituting on the right from Equations 27 and 29, we deduce

$$\mathrm{d}\varepsilon = [r^{-1} + n'(r)/n(r)]\cos\varepsilon\, \mathrm{d}s. \tag{32}$$

Equation 32 is the desired equation relating dε to ds.

In summary, if current values of the state variables r, θ and ε of the ray are known, then displacement along the ray by an infinitesimal distance ds will change these state variables by the respective infinitesimal amounts [cf. Equations 26, 27, and 32]

$$\begin{cases} dr = \sin\varepsilon \, ds, & (33a) \\ d\theta = r^{-1} \cos\varepsilon \, ds, & (33b) \\ d\varepsilon = [r^{-1} + n'(r)/n(r)] \cos\varepsilon \, ds. & (33c) \end{cases}$$

Equations 33 constitute the main result of our analysis. They imply that r, θ, and ε can be regarded as functions of the ray length s, which functions may be found by solving the set of coupled, first-order, ordinary differential equations

$$\begin{cases} dr/ds = \sin\varepsilon, & (34a) \\ d\theta/ds = r^{-1} \cos\varepsilon, & (34b) \\ d\varepsilon/ds = [r^{-1} + n'(r)/n(r)] \cos\varepsilon, & (34c) \end{cases}$$

subject to the specified initial conditions $r = r_0$, $\theta = \theta_0$ and $\varepsilon = \varepsilon_0$ at $s = 0$.

Actually, we can write down three more equivalent sets of differential equations by successively taking r, θ and ε as the independent variable instead of s. For example, if we solve Equations 33 for dr, ds, and dε in terms of dθ, we may deduce the set of differential equations

$$\begin{cases} dr/d\theta = r^{-1} \tan\varepsilon, & (35a) \\ ds/d\theta = r/\cos\varepsilon, & (35b) \\ d\varepsilon/d\theta = 1 + r^{-1} n'(r)/n(r). & (35c) \end{cases}$$

The advantage of using s as the independent variable is that the resulting differential equations 34 are well behaved for r bounded away from zero. By contrast, for example, Equations 35a and 35b become unbounded as $\varepsilon \rightarrow \pm \pi/2$.

SOLVING THE RAY EQUATIONS

Keeping in mind that *the ray equations 33 and their associated o.d.e. forms are approximately valid only for light rays of sufficiently high frequency*, let us now consider how to go about solving these equations.

Case $n(r)$ = Constant

If $n(r)$ is independent of r, then the second term in brackets in Equation 33c vanishes, and we have

$$d\varepsilon = r^{-1} \cos\varepsilon \, ds = d\theta, \qquad (36)$$

where the second equality has invoked Equation 33b. Integrating, we get

$$\varepsilon - \varepsilon_0 = \theta - \theta_0. \tag{37}$$

Now dividing Equation 33a by Equation 33b, and then substituting for ε from Equation 37, we get

$$\frac{dr}{d\theta} = \frac{r\sin(\theta - \theta_0 + \varepsilon_0)}{\cos(\theta - \theta_0 + \varepsilon_0)},$$

or

$$\frac{dr}{r} = -\frac{d\cos(\theta - \theta_0 + \varepsilon_0)}{\cos(\theta - \theta_0 + \varepsilon_0)}. \tag{38}$$

Equation 38 is easily integrated to give

$$\ln\left(\frac{r}{r_0}\right) = -\ln\left(\frac{\cos(\theta - \theta_0 + \varepsilon_0)}{\cos(\theta_0 - \theta_0 + \varepsilon_0)}\right) = \ln\left(\frac{\cos\varepsilon_0}{\cos(\theta - \theta_0 + \varepsilon_0)}\right),$$

whence

$$r\cos(\theta - \theta_0 + \varepsilon_0) = r_0\cos\varepsilon_0. \tag{39}$$

Equation 39 is just the polar equation of a *straight line*, which is precisely the ray equation we expect when the index of refraction is constant.

Case $n(r) = A/r$.

If $n(r) = A/r$, where A is a constant, then we have $n'(r) = -A/r^2$, so that

$$n'(r)/n(r) = (-A/r^2)/(A/r) = -r^{-1}. \tag{40}$$

In that case, the right side of Equation 33c vanishes everywhere, so that Equation 33c integrates to

$$\varepsilon = \varepsilon_0. \tag{41}$$

Thus, the trajectory is such that the elevation angle of the unit ray vector stays constant. If $\varepsilon_0 = 0$, then Equations 34a and 34b are easily integrated to give $r = r_0$ and $\theta = \theta_0 + s/r_0$, which is the equation of a *circle*. If $\varepsilon_0 \neq 0$, then integration of Equations 34a and 34b give

$$r = r_0 + s\sin\varepsilon_0, \tag{42a}$$

$$\theta = \theta_0 + \cot\varepsilon_0 \log[1 + (s/r_0)\sin\varepsilon_0]. \tag{42b}$$

These equations describe a curve that spirals either inward or outward, according to whether ε_0 is negative or positive.

Numerical Integrations

For even moderately realistic functions $n(r)$, usually we must proceed with a computer-oriented numerical calculation. The simplest, though *not* the most efficient, method is to recursively apply Equations 33 with the differentials dr, dθ, dε, and ds replaced by "small but finite increments" Δr, $\Delta\theta$, $\Delta\varepsilon$, and Δs, respectively. Figure 5 summarizes this method of constructing the ray; essentially, it is the *Euler method* of integrating the differential equations 34, subject to the initial conditions $r = r_0$, $\theta = \theta_0$, $\varepsilon = \varepsilon_0$ at $s = 0$. The key here is to choose Δs so small that reducing it further does not substantially alter the results. Of course, the smaller Δs is, the longer it will take to trace the ray through a given length s. The stopping condition in step 5 of Figure 5 will generally be that one of the variables r, θ, ε, or s has reached some predetermined value; or, it might be that a predetermined value has been reached by the *travel time t* along the ray, the increment of which is given by

$$\Delta t = (n(r)/c)\,\Delta s, \tag{43}$$

since $c/n(r)$ is the wave velocity at **r**.

A more efficient way of solving the set of differential equations (34) would be to simply use a packaged computer code. An example is the IMSL subroutine DVERK, which is available at most computer centers. Subroutine DVERK uses a high-order *Runge-Kutta* method, which yields a much more accurate estimate of the solution functions $r(s)$, $\theta(s)$, and $\varepsilon(s)$ for a given step size Δs, as compared to the simpler Euler method of Figure 5.

Regardless of what numerical method is selected to integrate Equations 34, a key user input is the index of refraction function, $n(r)$. At each Δs step the logarithmic derivative $n'(r)/n(r)$ must be computed anew, either from some assumed formula or else from tabulated data.

A strong *consistency check* on *any* numerical integration of Equations 34 can be had by periodically testing how well the conservation law in Equation 44 is satisfied.

A Conservation Law

We shall now show that Equations 33 imply that

$$r\,n(r)\cos\varepsilon = r_0\,n(r_0)\cos\varepsilon_0. \tag{44}$$

In other words, the quantity on the left side of Equation 44 is *constant* along any ray trajectory.

Dividing Equation 33c by 33a, we obtain

$$\frac{d\varepsilon}{dr} = \frac{[r^{-1} + n'(r)/n(r)]\cos\varepsilon}{\sin\varepsilon}.$$

Rearranging, we get

$$\frac{\sin\varepsilon\, d\varepsilon}{\cos\varepsilon} = \frac{dr}{r} + \frac{n'(r)\, dr}{n(r)} ,$$

or equivalently,

$$-\frac{d(\cos\varepsilon)}{\cos\varepsilon} = \frac{dr}{r} + \frac{d(n(r))}{n(r)} .$$

Integrating gives

$$-\ln\left(\frac{\cos\varepsilon}{\cos\varepsilon_0}\right) = \ln\left(\frac{r}{r_0}\right) + \ln\left(\frac{n(r)}{n(r_0)}\right).$$

This last equation can also be written

$$\ln\left(\frac{r\, n(r)\cos\varepsilon}{r_0 n(r_0)\cos\varepsilon_0}\right) = 0 \equiv \ln(1),$$

from which Equation 44 immediately follows.

Equation 44 can be viewed as a sort of generalization of *Snell's law*. The latter states that if a light ray crosses a planar surface at which the index of refraction changes *discontinuously* from n_1 to n_2, then the angles ε_1 and ε_2 between the surface and the rays on either side satisfy the equation $n_1\cos\varepsilon_1 = n_2\cos\varepsilon_2$. Although this last equation can in fact be derived from Equation 44 by using a limiting argument, one *cannot* legitimately proceed the other way around and derive the conservation law, Equation 44, or the ray equations (33), from Snell's law instead of from Maxwell's equations. The reason for this is that Snell's law applies to *plane* waves of *any* frequency at a planar *discontinuity* in the index of refraction, whereas the conservation law, Equation 44 and ray equations (33) apply to *distorted* plane waves of *high frequency only* in a medium where the index of refraction is a *continuous* (indeed, *differentiable*) function of r.

"Flat Earth" Formulas

It often happens that we are interested in ray trajectories in the Earth's atmosphere over distances that are extremely small compared to the Earth's radius R_E. In that case it is often permissible to assume that the ray is propagating in an xz plane, with the z axis pointing "up" and the x axis "horizontal" (and in the plane of the z axis and the initial unit ray vector). The r-only dependence of the index of refraction n easily translates into a z-only dependence. As shown in Figure 6, for sufficiently small θ we can define x and z in terms of r and θ by

$$z \approx r - R_E, \text{ and } x \approx r\sin\theta.$$

It follows that

$$dz \approx dr, \tag{45a}$$

and

$$dx \approx dr \sin\theta + r \cos\theta \, d\theta,$$

or, since $\theta \approx 0$ for the relatively small propagation distances of interest,

$$dx \approx r \, d\theta. \tag{45b}$$

Because of Equations 45, we may replace dr in Equation 33a with dz, and rdθ in Equation 33b with dx. Furthermore, we may assume that R_E is so large that $r^{-1} = (R_E + z)^{-1}$ is negligible compared to $n'(r)/n(r) = n'(z)/n(z)$ in Equation 33c. We thus conclude that the equations describing light ray trajectories in a "flat Earth" atmosphere, in which the index of refraction n is a function only of the altitude z, are:

$$\begin{cases} dz = \sin\varepsilon \, ds, & (46a) \\ dx = \cos\varepsilon \, ds, & (46b) \\ d\varepsilon = [n'(z)/n(z)] \cos\varepsilon \, ds. & (46c) \end{cases}$$

Notice that Equations 46a,b imply that $dz/dx = \tan\varepsilon$, showing that the elevation angle ε is just the conventional "slope angle" of the ray trajectory in the xz plane.

As with Equations 33, one can proceed to write down from Equations 46 four different sets of three coupled differential equations, one set for each choice of z, x, ε, or s as the independent variable. And also as before, the set with s as the independent variable is usually the best behaved.

The conservation law in Equation 44 now reads

$$(z + R_E)\, n(z) \cos\varepsilon = \text{constant}.$$

But since $R_E \gg z$, we can drop the z from the lead factor and then absorb R_E into the constant on the right. Thus, the "flat Earth" conservation law reads

$$n(z) \cos\varepsilon = n(z_0) \cos\varepsilon_0. \tag{47}$$

Indeed, it is easy to derive Equation 47 directly from Equations 46: We merely divide Equation 46c by Equation 46a, separate variables, and then integrate, just as we did in deriving Equation 44 from Equations 33. Equation 47 can evidently be used as a *consistency check* on any numerical integration of Equations 46.

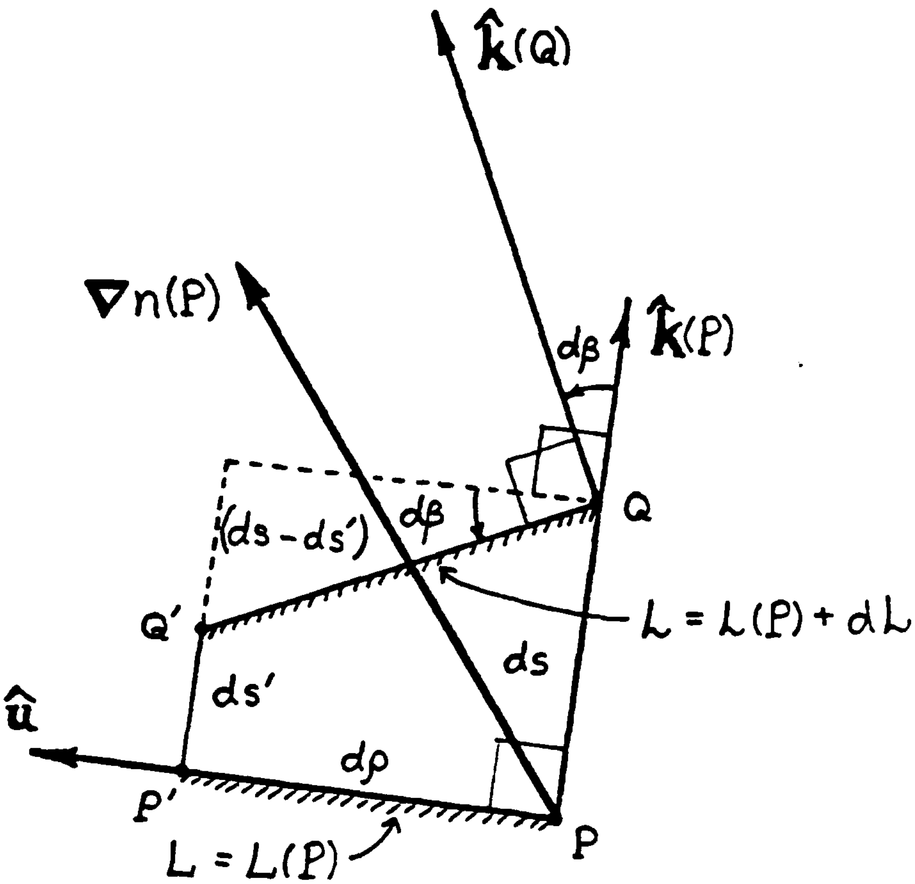

FIGURE 1. Geometry for deriving from the eikonal equation an expression for the infinitesimal angle $d\beta$ through which the unit ray vector $\hat{k}$ rotates in moving an infinitesimal distance ds along the ray.

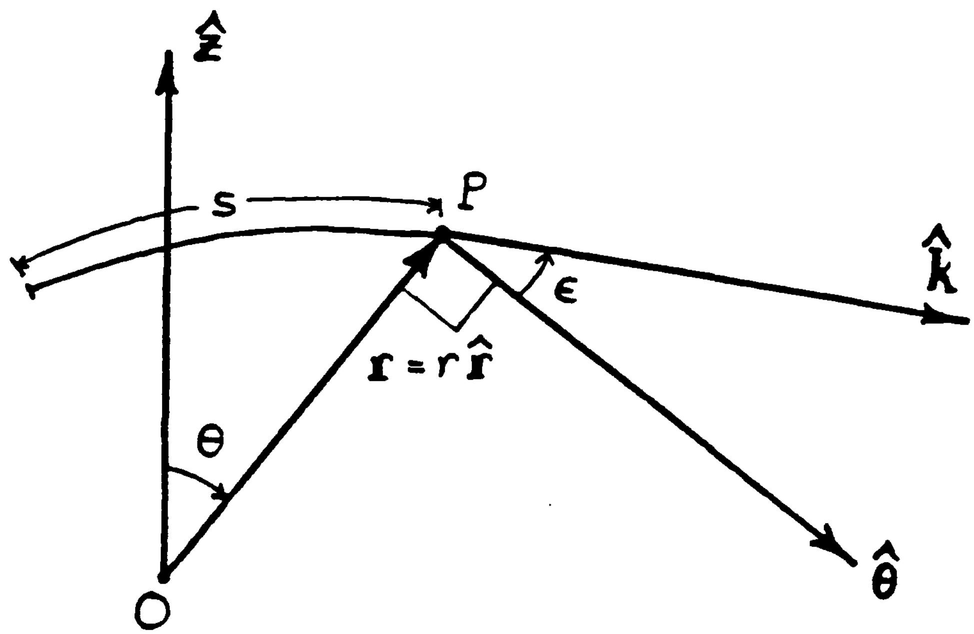

FIGURE 2. Geometry showing the variables r, θ and ε that define the instantaneous state of a ray passing through a point P in the direction $\hat{k}$. The origin of s, the path length along the ray, is arbitrary.

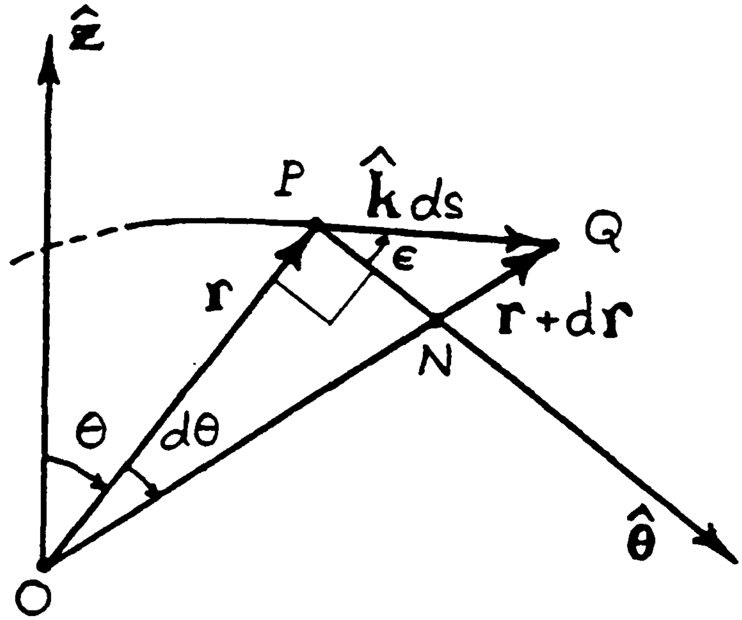

FIGURE 3. Geometry for deducing dr and $d\theta$ in terms of an infinitesimal displacement ds along the ray through P.

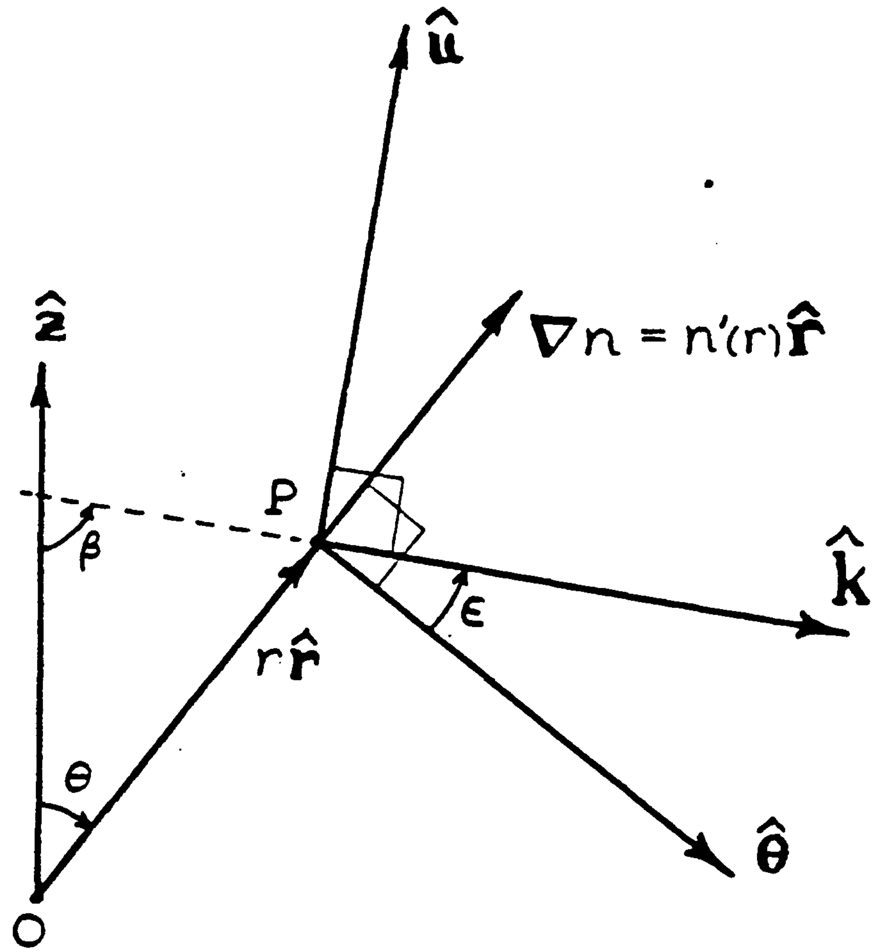

FIGURE 4. Geometry for calculating $d\varepsilon$ from the turning rate equation when $n(\mathbf{r}) = n(r)$.

1. Initialize: Set $r = r_0$, $\theta = \theta_0$, $\varepsilon = \varepsilon_0$ and $s = 0$.

2. Choose a small, positive value for Δs.

3. Calculate, for the current values of r and ε, the quantities:

$$\begin{cases} \Delta r = \sin\varepsilon \cdot \Delta s, \\ \Delta\theta = r^{-1}\cos\varepsilon \cdot \Delta s, \\ \Delta\varepsilon = [r^{-1} + n'(r)/n(r)]\cos\varepsilon \cdot \Delta s. \end{cases}$$

4. Update:

$$\begin{cases} r = r + \Delta r, \\ \theta = \theta + \Delta\theta, \\ \varepsilon = \varepsilon + \Delta\varepsilon, \\ s = s + \Delta s. \end{cases}$$

5. Record the current values of r, θ, ε, and s. Stop if the termination condition is satisfied: otherwise, go to Step 3.

FIGURE 5. The simple Euler method for numerically integrating Equations 34, given that at $s = 0$ we have $r = r_0$, $\theta = \theta_0$ and $\varepsilon = \varepsilon_0$. In Step 2, Δs should be chosen small enough that a repeat run with Δs chosen to be only half as large should not signifiently alter the terminal values of the variables.

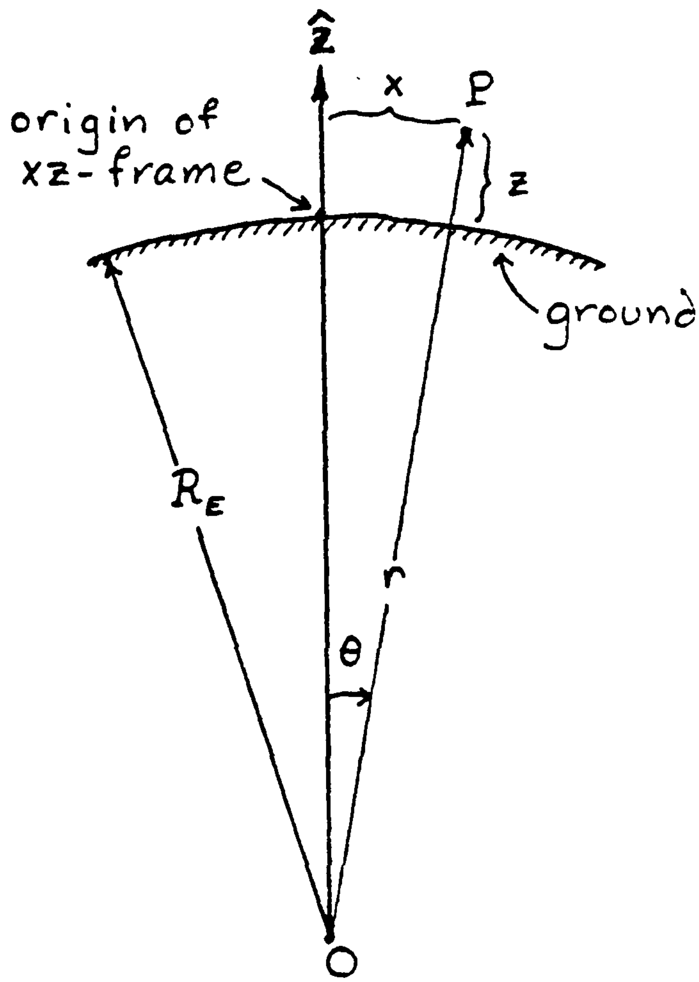

FIGURE 6. Geometry for deducing the form of the ray equations in Earth's atmosphere in situations where the curvature of the Earth's surface is negligible.

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